Logarithmic scaling of Lyapunov exponents in disordered chiral two-dimensional lattices

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We analyze the scaling behavior of the two smallest Lyapunov exponents for electrons propagating on two-dimensional lattices with energies within a very narrow interval around the chiral critical point at $E=0$ in the presence of a perpendicular random magnetic flux. By a numerical analysis of the energy and size dependence we confirm that the two smallest Lyapunov exponents are functions of a single parameter. The latter is given by $\ln L/\ln \xi(E)$, which is the ratio of the logarithm of the system width L to the logarithm of the correlation length $\xi(E)$. Close to the chiral critical point and energy $|E| \ll E_0$, we find a logarithmically divergent energy dependence $\ln \xi(E) \propto |\ln(E_0/|E|)|^{1/2}$, where E_0 is a characteristic energy scale. Our data are in agreement with the theoretical prediction of Fabrizio and Castelliani [Nucl. Phys. B **583**, 542 (2000)] and resolve an inconsistency of previous numerical work.

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I. INTRODUCTION

The numerical determination of transport parameters for electrons propagating in disordered two-dimensional (2D) systems with chiral symmetry still remains an important unsolved problem. The situation can be represented by a single-band tight-binding model defined on bipartite lattices subjected to purely off-diagonal disorder like a random magnetic flux with zero mean or real random hopping terms. The latter belongs to the chiral orthogonal universality class while the former is chiral unitary. Due to the chiral symmetry, the model exhibits metallic behavior only at energy $E=0$.[1] It is therefore of considerable interest to study the critical properties of the model in the vicinity of the critical point and to investigate its universality.

The traditional finite-size-scaling analysis of disorder driven metal-insulator transitions, i.e., continuous quantum phase transitions at zero temperature, is based on two assumptions.[2–5] (i) In the vicinity of the critical point all variables of interest are a function of only one parameter,

$$\upsilon(E,L) = F[L/\xi(E)], \tag{1}$$

where $\xi(E)$ is the energy-dependent correlation length. Here, vicinity means that both the system size L and $\xi(E)$ are already larger than any other typical length of the model and $\xi > L$. (ii) At the critical energy E_c, the correlation length $\xi(E)$ diverges as

$$\xi(E) \sim |E - E_c|^{-\nu} \tag{2}$$

with a universal critical exponent ν. Relation (2) was confirmed in a multitude of numerical work on disordered systems in spatial dimension $2 \le d \le 5$ and various physical symmetries.[6] As scaling variables, for example, the localization length,[7] the smallest Lyapunov exponent,[8] the two-terminal conductance,[9,10] the energy-level spacings,[11–13] and the inverse participation ratio[14] were successfully used.

Recently, the analysis of numerical data obtained for the energy dependence of the two-terminal conductance g on a bricklayer lattice,[15] which represents a generic lattice model for graphene, has led to a power-law energy dependence of the correlation length $\xi(E) \propto |E|^{-\nu}$, where the critical exponent ν is close to 1/3. Although this outcome is in agreement with previous numerical results[16–20] for square and hexagonal lattices, it is at variance with the Harris criterion[21] which states that $\nu > 1/d$, where $d=2$ is the Euclidian dimension of the system. More importantly, all numerical data obtained to the present date do not agree with theoretical predictions,[1,22] according to which the correlation length depends logarithmically on the energy,

$$\xi(E) = \xi_0 \exp[A\sqrt{\ln(E_0/|E|)}], \quad |E| \ll E_0, \tag{3}$$

where A is related to the longitudinal conductivity and E_0 is assumed to be of the order of the energy bandwidth.[1] A possible explanation of this disagreement between theory and numerical experiments may be that the energies investigated in the numerical studies, down to 10^{-10} so far[20] (in units of the hopping energy), are not sufficiently small in comparison to the unspecified parameter E_0 introduced in the theory. Thus, it could be that the energy interval $|E| \ll E_0$, where the scaling holds, was not reached in previous numerical studies. A second obstacle is the vanishing of the density of states $\rho(E)$ which occurs at $E=0$ for hexagonal and bricklayer lattices in the presence of random-magnetic-flux disorder.[23] This behavior persists even in strongly disordered chiral systems so that the two-terminal conductance, which nevertheless turns out to be finite $\sim e^2/h$ at the Dirac point in graphene,[15,24–26] is not a suitable scaling variable for numerical studies. Therefore, it is expedient to investigate instead the smallest Lyapunov exponents, which are associated with the localization length and are not directly affected by the vanishing density of states.

In this paper, we analyze the scaling behavior of both the two-dimensional bricklayer and square-lattice model with random-magnetic-flux disorder. Using the transfer-matrix method for quasi-one-dimensional (1D) systems,[3–5,8] we calculate the two smallest Lyapunov exponents z_1 and z_2 for energies very close to $E=0$, not achieved in previous work. Lyapunov exponents are more suitable than the conductance since they are less sensitive to the energy dependence of the

density of states. Another reason for using Lyapunov exponents is that the analysis of the conductance is far more time consuming, which is crucial since quadruple precision is necessary in our case when energies smaller than 10^{-16} are considered. We show that in the vicinity of the critical point our numerical data for the Lyapunov exponents lead to the relation

$$z_{1,2}(E,L) = F_{1,2}\left[\frac{\ln L}{\ln \xi(E)}\right], \tag{4}$$

where L is the width of the system. We prove that the correlation length $\xi(E)$ depends logarithmically on the energy [see Eq. (3)] in agreement with the predictions by Fabrizio and Castelliani.[22] Thus, our results resolve the previous discrepancy between analytical theory and numerical calculations.

II. MODEL AND METHOD

We study a single-band tight-binding Hamiltonian defined on a two-dimensional square lattice with nearest-neighbor hopping and random-flux disorder, which is introduced by complex phase factors in the transfer terms,

$$\mathcal{H}/V = \sum_{x,y}{}' (e^{i\theta_{x,y+a;x,y}} c^{\dagger}_{x,y} c_{x,y+a} + e^{-i\theta_{x,y-a;x,y}} c^{\dagger}_{x,y} c_{x,y-a}) + \sum_{x,y} (c^{\dagger}_{x,y} c_{x+a,y} + c^{\dagger}_{x,y} c_{x-a,y}), \tag{5}$$

where $c^{\dagger}_{x,y}$ and $c_{x,y}$ denote creation and annihilation operators of a fermionic particle at site (x,y), respectively. For bricklayer lattices, the prime at the first sum in Eq. (5) indicates that only the transfers along every other vertical bond are included. In this way, the square lattice is transformed into a bricklayer where the coordination number is reduced to three nearest-neighbor sites. The bricklayer lattice has the same topology as the honeycomb lattice of graphene and Hamiltonian (5) possesses the same eigenvalues $\pm\varepsilon_i$. The phases, which are chosen to be associated only with the vertical bonds in the y direction, $\theta_{x,y;x,y+a} = \theta_{x+2a,y;x+2a,y+a} - \frac{2\pi e}{h}\phi_{x,y}$, are defined by the random magnetic flux $\phi_{x,y}$, which is uniformly distributed $-f/2 \le \phi_{x,y} \le f/2$ with zero mean and disorder strength $0 \le f/(h/e) \le 1$. The random magnetic flux is pointing perpendicular to the 2D lattice and periodic boundary conditions are applied in the y direction. In contrast to diagonal disorder, this random flux preserves the chiral symmetry for both the square and bricklayer lattices. We fix the units of energy and length scales by the nearest-neighbor hopping energy $V=1$ and the lattice constant $a=1$, respectively. The disorder strength is taken to be $f=0.5h/e$ for the bricklayer and $f=1.0h/e$ for the square lattice.

We use the transfer-matrix method[6] and collect numerical data for the two smallest Lyapunov exponents $z_1(E,L)$ and $z_2(E,L)$. For the system width L and length $L_x \gg L$, we calculate the transfer matrix $\mathbf{M} = \Pi_i^{L_x} \mathbf{M}_i$ and extract the two smallest Lyapunov exponents. The relative uncertainty $\epsilon(E,L)$ of our data is 2×10^{-3} for larger widths $L=192$ and $L=160$, and decreases down to 10^{-4} for the smallest $L=8$. This requires the length of the quasi-1D systems L_x to be in the range $\sim 10^8 - 10^9$. Since we expect that scaling occurs only in the vicinity of the $E=0$ critical point, we consider energies as small as possible, down to the point of $|E| = 10^{-34}$, at least for $L \le 64$. This requires to perform the calculations with quadruple numerical precision.

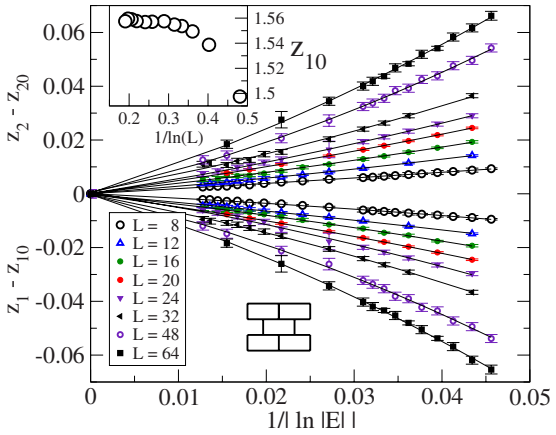

FIG. 1. (Color online) The smallest Lyapunov exponents $z_1(E,L) - z_1(E=0,L)$ (lower branches) and $z_2(E,L) - z_2(E=0,L)$ (upper branches) as a function of $1/|\ln|E||$ for energies $|E| < 3\times10^{-10}$. The applied random-flux strength is $f=0.5h/e$ and the width of the quasi-1D systems is in the range $8 \le L \le 64$. Solid lines are quadratic fits. The inset shows the size dependence of $z_{10} = z_1(E=0,L)$ for $8 \le L \le 192$.

The specific symmetry of the model provides us with an independent test of the accuracy of our data. Due to the chiral symmetry, the spectrum of Lyapunov exponents must be degenerate at the band center for all L,

$$z_1(E=0,L) = z_2(E=0,L). \tag{6}$$

Deviations from $E=0$ remove this degeneracy but the average value, $[z_1(E)+z_2(E)]/2$, equals to $z_1(E=0)$ for small values of E.

III. BRICKLAYER LATTICE

The energy dependence of the two Lyapunov exponents is plotted in Fig. 1 for various system widths L of the bricklayer. Our data confirm that z_1 and z_2 are analytical functions of the variable $1/|\ln|E||$. Therefore, we approximate their energy dependence by the Taylor expansion

$$z_1(\chi,L) - z_1(0,L) = c_0(L) + c_1(L)\chi + c_2(L)\chi^2, \tag{7}$$

where

$$\chi = \frac{1}{|\ln(E_0/|E|)|} \tag{8}$$

and by a similar expansion $z_2(\chi,L) - z_2(0,L) = d_2(0,L) + d_1(L)\chi + d_2(L)\chi^2$, for the second Lyapunov exponent. Comparing with Eq. (2), we conclude that $\ln(\xi_0/a) \ll A[\ln(E_0/|E|)]^{1/2}$ so that in what follows we consider $\xi_0 \sim a$. The expansion coefficients c_i and d_i are determined numerically. The L dependence of the coefficient c_0 and d_0 shows finite-size corrections. For the bricklayer, we found that c_0 and d_0 depend only weakly on L provided that L

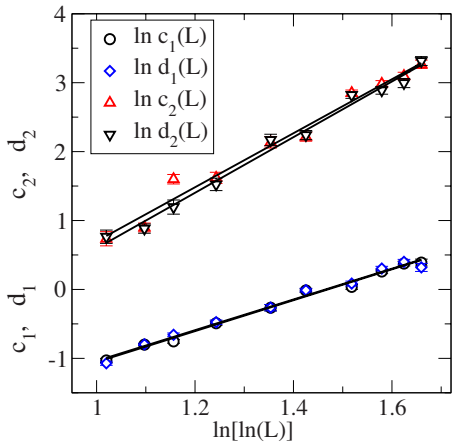

FIG. 2. (Color online) The size dependence of the coefficients c_i and d_i, given by Eqs. (11) and (12). Solid lines are linear fits with slopes 2.24 ± 0.26, 2.23 ± 0.26 (c_1 and d_1) and 3.92 ± 0.38, 4.03 ± 0.40 (c_2 and d_2). The width is in the range $16 \leq L \leq 192$ and $E_0 = 0.8$. Only data for $|E| \leq 3 \times 10^{-10}$ were used.

> 16 (data are shown in the inset of Fig. 1). For instance, we obtain that $z_1(E=0, L=16) = 1.5498 \pm 0.0003$ and $z_1(E=0, L=192) = 1.557 \pm 0.002$.

To estimate the energy E_0, we first minimize the expression

$$X = \sum_{E,L} \frac{[z(E,L) - F(E,L)]^2}{[z(E,L)\epsilon(E,L)]^2}, \tag{9}$$

where

$$F(E,L) = \beta_0 + \beta_1 (\ln L)^{\alpha_1} \chi + \beta_2 (\ln L)^{\alpha_2} \chi^2 \tag{10}$$

with respect to parameters α, β, and E_0. We found that X possesses a minimum when $2.2 < \alpha_1 < 2.5$, $3.8 < \alpha_2 < 4.1$, and $0.1 < E_0$. It was not possible to obtain a better estimation of the critical parameters from this procedure since small variation in E_0 can be compensated by small change in β_2 and α_2. Instead, in a more accurate analysis, we fit our numerical data for z_1 and z_2 to the quadratic expansion [Eq. (7)]. Figure 2 shows the L dependence of the coefficients $c_{1,2}$ and $d_{1,2}$. Our data confirm the assumed logarithmic behavior of all coefficients occurring in the expansion

$$c_1(L), \quad d_1(L) \propto [\ln(L)]^{\alpha_1} \tag{11}$$

and

$$c_2(L), \quad d_2(L) \propto [\ln(L)]^{\alpha_2}, \tag{12}$$

where α_1 and α_2 are close to the anticipated values 2 and 4, respectively.

Figure 3 shows another test of the scaling of the Lyapunov exponents. Following the conventional scaling method,[5] we rescaled the horizontal axis for the data shown in Fig. 1 by the parameter $C(L)$: $\chi \to \chi C(L)$. The such obtained $C(L)$ gives us directly the required scaling behavior as shown in the Fig. 3. The data for both z_1 and z_2 scale to one universal curve. The inset to Fig. 3 confirms the expected power-law relation $C(L) \propto (\ln L)^2$.

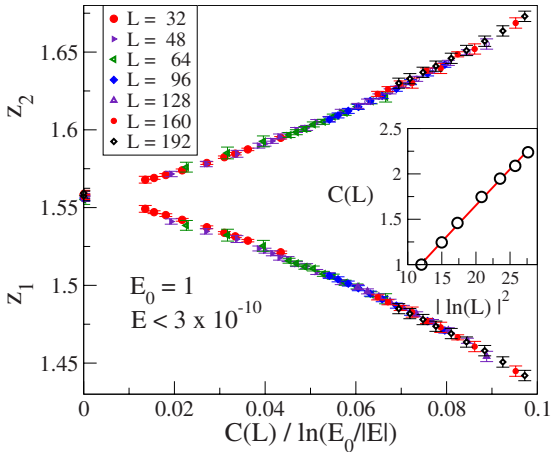

FIG. 3. (Color online) Finite-size scaling of the two lowest Lyapunov exponents for various widths L and restricted energies $|E| < 3 \times 10^{-10}$. The inset shows the L dependence of the parameter $C(L)$. The solid line is the power-law fit confirming that $C(L) \propto (\ln L)^{1.934}$.

To obtain a quantitative estimation of the energy E_0, we repeated the scaling analysis shown in Fig. 3 for various E_0. Although we recovered the scaling behavior similar to that shown in Fig. 1 (data not shown), the L dependence of the parameter $C(L)$ depends on the choice of E_0. As shown in Fig. 4, $C(L) \propto (\ln L)^{\kappa}$ with the exponent κ decreasing when E_0 increases, converging to $\kappa \approx 2$ for $E_0 \gtrsim 1$. We conclude that the energy E_0 is on the order of unity in our bricklayer model.

Finally, we plot the two smallest Lyapunov exponents as a function of a single parameter $(\ln L)^2 / \ln(E_0/|E|)$ with $E_0 = 0.8$ in Fig. 5. To reduce the finite-size corrections, we subtract from the data the values $z_1(E=0.L)$ and $z_2(E=0,L)$, respectively. All data collapse onto a single curve.

IV. SQUARE LATTICE

Another possibility to check for logarithmic scaling at chiral quantum critical points is the numerical analysis of a

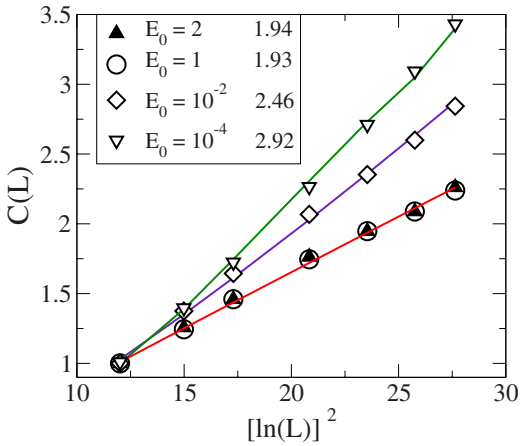

FIG. 4. (Color online) The function $C(L)$ obtained by a scaling analysis of the data for four different values of the energy E_0. Solid lines are power-law fits $C(L) \propto (\ln L)^{\kappa}$ with the value of the exponent κ given in the legend.

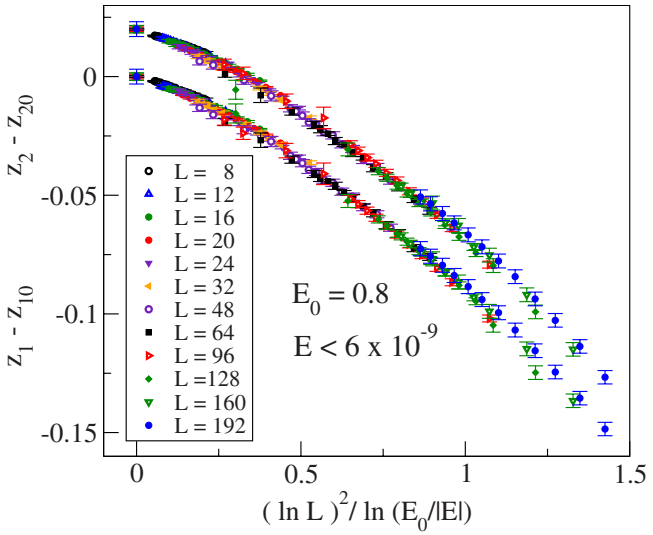

FIG. 5. (Color online) The data collapse of the first two Lyapunov exponents $z_1(E,L)-z_{10}(L)$ and $z_2(E,L)-z_{20}(L)$ plotted as a function of a single parameter $(\ln L)^2/\ln(E_0/|E|)$ with $E_0=0.8$. For clarity, the data for the second Lyapunov exponent are shifted vertically.

simple square lattice.[16–20] We calculated $z_1(E,L)$ and $z_2(E,L)$ for L even and Dirichlet boundary conditions in the transverse direction. We found that the scaling analysis is more difficult than for the bricklayer. First, the finite-size effects are more pronounced (see lower inset in Fig. 6). We can eliminate them, at least partially, by subtracting the value $z(E=0,L)$ from $z(E,L)$.[27] Second, in the unperturbed model

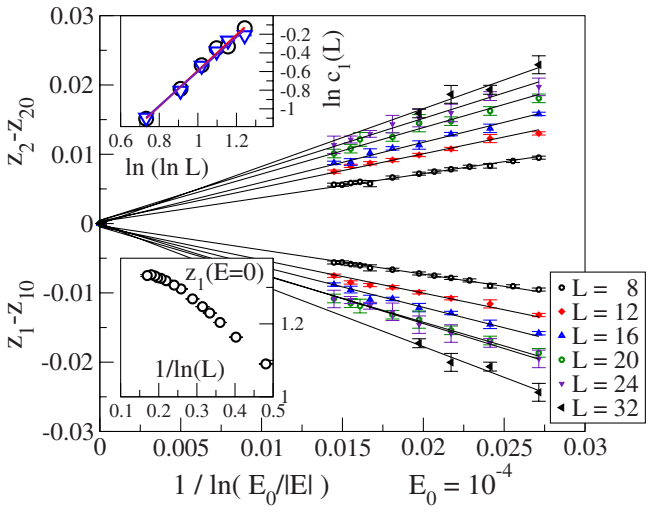

FIG. 6. (Color online) The energy dependence of the two smallest Lyapunov exponents $z_1(E)-z_1(E=0)$ and $z_2-z_2(E=0)$ calculated for the square lattice. The applied random-flux strength is $f=1.0h/e$. Owing to the small value of E_0, we restricted the energy interval to $|E|\leq 10^{-20}$. Then, both z_1 and z_2 are linear functions of χ. The first inset shows the L dependence of the coefficient c_1 of the Taylor expansion [Eq. (7)]. The exponents α_1 obtained are 1.92 ± 0.24 for z_1 and 1.85 ± 0.25 for z_2. The plot of $z_1(E=0,L)$ vs $1/\ln L$ demonstrates the finite-size effects which are much stronger than for the brick layer (lower inset).

the van Hove singularity, which appears at $E=0$ compared with $E=\pm1$ for the bricklayer, may spoil the scaling analysis. More importantly, following the same procedure as for the bricklayer, we found that the function X given by Eq. (9) possesses a minimum only for small $E_0\sim10^{-4}$ although the energy bandwidths are about the same for both lattices. Since the energy E must be much smaller than E_0, we had to restrict our analysis to energies $|E|\leq10^{-20}$. Fortunately, in such a narrow energy interval, we can neglect the quadratic term in the Taylor expansion [Eq. (7)]. As shown in Fig. 6, both z_1 and z_2 are linear functions of χ when $L\leq32$. This enables us to estimate the exponent α_1 from the analysis of the size dependence of the slope, $c_1(L)\propto(\ln L)^{\alpha_1}$. This analysis is independent on both the choice of E_0 and finite-size effects, provided that the latter do not depend on the energy.

However, the fit turns out to be rather unstable to small changes in the data ensemble. First, the interval of χ is very narrow and almost all data points are accumulated in the right part of this interval. Therefore, the resulting fit is very sensitive to the exact value of $z_1(E=0)$. Second, although we calculated our data with high accuracy, Fig. 6 shows that this is still not sufficient for a perfect determination of the slope. To check the accuracy of α_1, we tested various data ensembles and found that α_1 varies between 1.7 and 2.1. Nevertheless, our data for the square lattice are compatible with a logarithmic scaling relation.

V. CONCLUSIONS

We analyzed the scaling behavior of the two smallest Lyapunov exponents z_1 and z_2 in disordered two-dimensional chiral systems defined on a bricklayer and on a square lattice. We found that both z_1 and z_2 follow a logarithmic scaling relation as considered by Sittler and Hinrichsen[28] with a correlation length proposed by Fabrizio and Castelliani.[22] According to Ref. 28, the physical origin of logarithmic scaling is associated with multifractality and local scaling invariance. The results presented above are the numerical confirmation of the scaling relation (4) in which the scaling variable is given by the ratio of the logarithm of the system width L to the logarithm of the correlation length ξ, instead of the ratio L/ξ as applied usually. This scaling is accompanied by the logarithmic energy dependence of the correlation length $\ln\xi(E)\propto\sqrt{\ln(E_0/|E|)}$ valid for $|E|\ll E_0$. Our results also solve the contradiction between previous numerical work, which apparently did not reach the true scaling regime, and the Harris criterion.

Two methods of the scaling analysis were used. Both confirm that the logarithmic scaling is observable only for very small values of the energy close to the chiral quantum critical point at $E=0$. In order to resolve the Lyapunov exponents for energies down to $|E|=10^{-34}$, the implementation of quadruple precision in the numerical algorithms was necessary. This probably explains why logarithmic scaling was not observed in previous numerical work.[15–20]

The question arises whether the same logarithmic scaling analysis can be performed also for the two-terminal conductance. At present this seems not possible with our available computing power. In our previous work,[15] we found

$g(E,L)=g_0 \ln[\tilde{E}^\star(L)/|E|]$ for $|E|>E^\star=\tilde{E}^\star/\text{const}$, but did not observe any energy and system size dependence of the ensemble averaged conductance $g_c \simeq 4/\pi e^2/h$ as long as the energy remains smaller than a certain value $E^\star \propto L^{-2}$. This size-dependent energy interval coincides with the recently observed depression in the density of states.[23] The observation of a tiny logarithmic energy dependence of the conductance, if present at all, would require a far more accurate numerical determination of the ensemble averaged mean conductance.

As shown analytically in Ref. 22, the logarithmic energy dependence of the correlation length near $E=0$ is accompanied by a divergence of the density of states $\rho(E) \propto E^{-1} \exp[-(4A \ln E_0/E)^{1/2}]$. Such a relation is, however, not found in recent numerical work on a unitary chiral lattice model, where the density of states decreases to zero when $E \to 0$.[23] Also, a different divergency exponent of the density of states $\rho(E) \propto E^{-1} \exp[-1/2(c|\ln E/E_0|)^{2/3}]$ was derived analytically for the chiral orthogonal model.[29,30] This difference shows also up in the energy dependence of the correlation length. It would be very interesting to see, whether this subtle difference can also be observed in numerical studies on a bricklayer lattice with real random hopping disorder, which belongs to the chiral orthogonal symmetry class.

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